CH 41 – EXPONENTIAL FUNCTIONS

☐ INTRODUCTION

What worried Steve McQueen was not that The Blob was growing by a constant amount every hour, but rather that it was doubling in size every hour. If our hero had known his math, he could have warned the town that The Blob was not growing linearly, but *exponentially*.

☐ EXPONENTIAL FUNCTIONS

EXAMPLE 1: **Analyze The Blob function.**

Solution: To give you an idea of the concept of an exponential function, let's look at the two scenarios regarding the rate of growth of The Blob. An example of a linear rate of growth might be the formula $B = 4t$, where t is the number of hours and B is the size of The Blob. An exponential formula could be $B = 2^t$. If we construct a table, showing the time and the amount of The Blob for each formula, we can see the true effect of exponential growth.

t	1	2	3	4	5	6	7	8	9	10	11
$4t$	4	8	12	16	20	24	28	32	36	40	44
2^t	2	4	8	16	32	64	128	256	512	1024	2048

For the first three hours, there's more blob in the linear formula than in the exponential formula. At $t = 4$, the blob amounts are equal. But after that, it's not even a contest — the exponential formula shows that The Blob will probably eat the town, the state, and eventually the entire Earth!

EXAMPLE 2: **Find some ordered pairs for the exponential function $f(x) = 4^x$.**

Solution:

$$\begin{aligned}
f(1) &= 4^1 = 4 && \Rightarrow \quad (1, 4) \\
f(-2) &= 4^{-2} = \frac{1}{4^2} = \frac{1}{16} && \Rightarrow \quad (-2, \tfrac{1}{16}) \\
f(-1) &= 4^{-1} = \frac{1}{4} && \Rightarrow \quad (-1, \tfrac{1}{4}) \\
f(0) &= 4^0 = 1 && \Rightarrow \quad (0, 1) \\
f(\tfrac{1}{2}) &= 4^{1/2} = \sqrt{4} = 2 && \Rightarrow \quad (\tfrac{1}{2}, 2) \\
f(2) &= 4^2 = 16 && \Rightarrow \quad (2, 16)
\end{aligned}$$

We now try to determine exactly what the formula for an exponential function looks like, and how it differs from that of a polynomial. Look at the exponential functions given in the previous examples:

$$B = 2^t \qquad f(x) = 4^x$$

Notice that in each function, the base is a constant (the 2 and the 4) and the exponent is a variable (the t and the x). Thus, an ***exponential function*** is a function of the form

$$f(x) = \underset{\substack{\uparrow \\ \boxed{\text{constant}}}}{b}^{\,x} \leftarrow \boxed{\text{variable}}$$

where b is some appropriate real number (a constant)

This is in sharp contrast to the notion of a *polynomial*, which is the other way around. Thus,

$y = 10^x$ is an *exponential* function,
$y = x^{10}$ is a *polynomial* function, and
$y = x^x$ is neither an exponential nor a polynomial function.

The question of which real numbers b serve nicely as the base of an exponential function will be discussed later.

Homework

1. a. Fill in the following chart, similar to Example 1:

t	1	2	3	4	5	6	7	8	9	10
$9t$										
3^t										

b. Is the $9t$ row of the chart a linear or exponential function?

c. Is the 3^t row of the chart a linear or exponential function?

d. For how many hours does the linear growth produce more blob than the exponential growth?

e. At what hour do both growths give the same amount of blob?

f. At 10 hours, what is the ratio of the exponential amount of blob to the linear amount of blob?

2. Let's find some more ordered pairs in the function $f(x) = 4^x$ from Example 3.

a. (3, ___) b. (–3, ___) c. $(-\frac{1}{2}, \_\_\_)$ d. $(\frac{3}{2}, \_\_\_)$

3. Take a guess what the domain of $f(x) = 4^x$ is.

GRAPHING EXPONENTIAL FUNCTIONS

Let's use the function $y = 4^x$ discussed in the previous section to make our first exponential graph.

<u>EXAMPLE 3:</u> **Graph:** $y = 4^x$

<u>Solution:</u> Here are some ordered pairs for this function that we found in Example 2 and Homework #2:

$(-1, \frac{1}{4})$ $(-\frac{1}{2}, \frac{1}{2})$ $(0, 1)$ $(\frac{1}{2}, 2)$ $(1, 4)$ $(2, 16)$

Notice that the point (0, 1) is the ***y*-intercept**.

Next we analyze a pair of **limits**. First we'll let $x \to \infty$. As it does, the functional value 4^x approaches ∞ <u>much</u> faster than x does. For example, as x takes the values 6, 8, 10, the y-values go 4096, 65536, 1048576. Wow! The functional values are growing like crazy. The following limit should now be clear:

As $x \to \infty$, $y \to \infty$ [an amazingly fast increase]

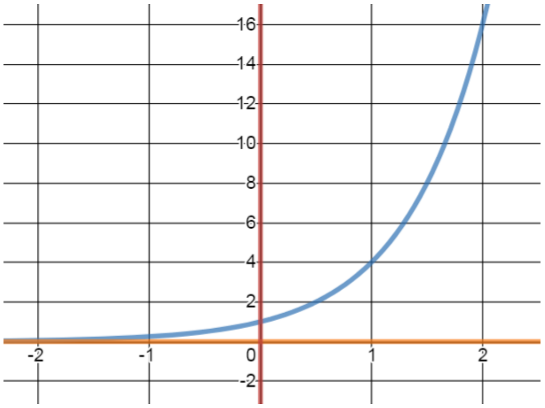
Second, we analyze what the y-values do when $x \to -\infty$. Consider the three ordered pairs:

(–5, 0.000977) (–8, 0.000015) (–10, 0.000000954)

It appears that as x grows smaller (towards $-\infty$), the y-values are positive numbers shrinking toward zero. That is,

As $x \to -\infty$, $y \to 0$

The ordered pairs we listed and the limits we calculated lead us to the following graph:



The **domain** is ℝ. Also, there is no vertical **asymptote**, but the line $y = 0$ (the x-axis) is a horizontal asymptote.

Last, it appears that there is no ***x*-intercept** on our graph. Even more importantly, this confirms that

The equation $4^x = 0$ has <u>no</u> solution.

<u>EXAMPLE 4:</u> **Graph:** $y = \left(\frac{1}{2}\right)^x$

<u>Solution:</u> Let's get right to some ordered pairs.

If $x = -3$, then $y = \left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$, which gives us the ordered pair (–3, 8). It's now your job to verify each of the following ordered pairs in our function:

$(0, 1)$ $(1, \frac{1}{2})$ $(2, \frac{1}{4})$ $(3, \frac{1}{8})$ $(4, \frac{1}{16})$

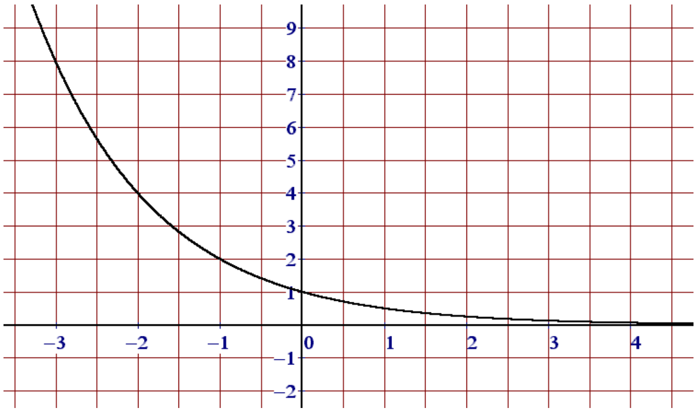
$(-1, 2)$ $(-2, 4)$ $(-3, 8)$ $(-4, 16)$

The y-intercept is (0, 1). There is no x-intercept, since the equation $\left(\frac{1}{2}\right)^x = 0$ has no solution. The ordered pairs listed above give credence to the following limits:

As $x \to \infty, y \to 0$ and As $x \to -\infty, y \to \infty$

The ordered pairs and the limits lead us to the following graph:

We can see that the domain of this function is $\mathbb{R}$. Notice also that we have a horizontal asymptote at $y = 0$, but there is no vertical asymptote.



EXAMPLE 5: **Graph: $f(x) = 2^{-x}$**

Solution: Two observations and we'll be done in a jiffy. First, we know that $f(x)$ can be written simply as y. Second, check out the following calculation:

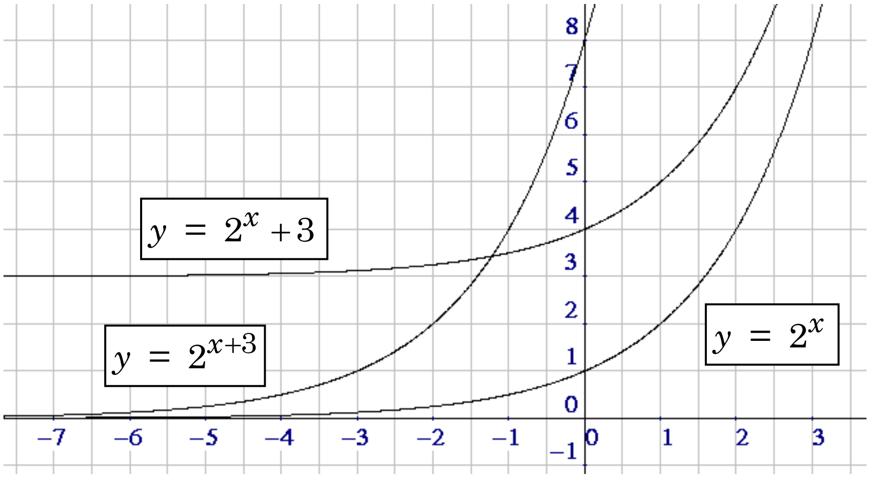
$$2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$$

In other words, the original formula can be written $y = \left(\frac{1}{2}\right)^x$, which we just finished graphing. So the solution to this problem is identical to that of the previous example.

EXAMPLE 6: **Graph: $y = 2^x$ and $y = 2^x + 3$ and $y = 2^{x+3}$.**

Solution: Let's make one table showing x and all the y-values at once and a single grid containing all the graphs at once.

x	2^x	$2^x + 3$	2^{x+3}
–6	1/64	3 1/64	1/8
–5	1/32	3 1/32	1/4
–4	1/16	3 1/16	1/2
–3	1/8	3 1/8	1
–2	1/4	3 1/4	2
–1	1/2	3 1/2	4
0	1	4	8
1	2	5	16
2	4	7	32
3	8	11	64



You should note that the graph of $2^x + 3$ is just the graph of 2^x but shifted 3 units up. Also, we see that the graph of 2^{x+3} is the result of taking the graph of 2^x and shifting it 3 units to the left.

Homework

4. Referring to Example 4, explain the last paragraph.

5. Graph: $f(x) = 3^x$

6. Graph: $y = 3^x - 2$

7. Graph: $y = 3^{x+2}$

8. Graph: $g(x) = \left(\frac{1}{3}\right)^x$

9. Graph: $h(x) = 3^{-x}$

☐ *THE LEGAL BASES OF AN EXPONENTIAL FUNCTION*

In the previous section we graphed exponential functions with bases 4, $\frac{1}{2}$, 2, 3, and $\frac{1}{3}$. Now it's time to figure out exactly which bases we'll allow in the exponential function

$$f(x) = b^x$$

Whatever values of b we allow to be the base of an exponential function, we'd like the domain of the function (the legal x-values) to be $\mathbb{R}$, the set of real numbers. And we don't want the exponential function to degenerate into some simple function that doesn't possess the "exponential" properties we've seen up til now.

$\boldsymbol{b < 0}$ What about negative bases? Consider $f(x) = (-4)^x$. If we choose $x = \frac{1}{2}$, the functional value is $(-4)^{1/2} = \sqrt{-4}$, not a real number. We thus disallow any base b that is negative.

$\boldsymbol{b = 0}$ Now consider $f(x) = 0^x$. But 0^x is fraught with problems. For example, if $x = 0$, we get 0^0. Can we assign a value to 0^0? On the one hand, 0 to any power should be 0. On the other hand, anything to the 0 power is supposed to be 1. So 0^0 is meaningless (but can be solved in Calculus II). Even worse, consider 0^{-2}. Since a negative exponent indicates reciprocal, we get $\frac{1}{0^2} = \frac{1}{0}$, which is undefined. All in all, a base of 0 really stinks.

$\boldsymbol{0 < b < 1}$ These bases are just fine. We used bases of $\frac{1}{2}$ and $\frac{1}{3}$ in the previous section. Even the number $\frac{1}{\pi}$ would be a legal base, although I've never seen it used.

$\boldsymbol{b = 1}$ This gives us the function $f(x) = 1^x$, which is the function $f(x) = 1$, a constant function (the horizontal line $y = 1$). Exponential functions aren't supposed to be flat, so b can't be 1.

$\boldsymbol{b > 1}$ Any base bigger than 1 is appropriate. In fact, in computer science a base of **2** is very popular. In basic science, the best base is **10** (for things like acids, earthquakes, and the volume of sound). And in calculus and the more advanced sciences, we use a number seen earlier: "***e***".

Homework

10. Describe precisely the legal bases for an exponential function.

11. Explain why –9 is not a good base for an exponential function.

12. Which of the following real numbers are legal bases for an exponential function?

-1	$-.01$	0	$\frac{2}{3}$	0.987	1	π	200

Practice Problems

13. T/F: $y = x^3$ is an exponential function.
14. Describe the real numbers which can be used as the base of an exponential function.
15. Explain why 1 is not a good base for an exponential function.
16. Give a function which is both an exponential function <u>and</u> a polynomial function.
17. Graph $y = 5^x$.
18. Graph $y = 3^{-x} - 2$ and state its horizontal asymptote.
19. Let $f(x) = 2^x$. Now let g be the graph which results from taking the graph of f and shifting it 7 units to the left and 4 units up. Find a formula for g.
20. T/F: $y = 3^x$ is an exponential function.
21. T/F: In the exponential function $f(x) = b^x$, b can be any positive real number.

22. What is the domain of the function $y = 10^{7x+1} - 10$?

23. Find all the asymptotes of the function $f(x) = 99^x$.

24. Explain why the graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = 3^{-x}$ are the same.

25. How does the graph of $f(x) = 5^{x-2} + 4$ compare with that of $y = 5^x$?

26. T/F: All exponential functions are increasing functions.

27. Explain why 0 is not a good base for an exponential function.

28. True/False:
    a. $y = x^3$ is an exponential function.
    b. $y = \pi^x$ is an exponential function.
    c. The domain of the function $y = 4^x$ is $[0, \infty)$.
    d. The function $y = x^x$ is neither polynomial nor exponential.
    e. The function $y = 2^x - 1$ has an x-intercept.
    f. For the function above, as $x \to -\infty$, $y \to 0$.
    g. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$. As $x \to \infty$, $y \to \infty$.
    h. Compared to the graph of $y = 4^x$, the graph of $y = 4^{x-5}$ is five units lower.
    i. Any real number $b > 0$ is a legal base for an exponential function.
    j. Any real number $b \geq 0$, but not equal to 1, is a legal base for an exponential function.
    k. The number $\pi + \sqrt{2}$ is a legal base for an exponential function.
    l. All exponential functions are decreasing functions.

Solutions

1. a.

t	1	2	3	4	5	6	7	8	9	10
$9t$	9	18	27	36	45	54	63	72	81	90
3^t	3	9	27	81	243	729	2187	6561	19683	59049

b. linear c. exponential d. first two hours

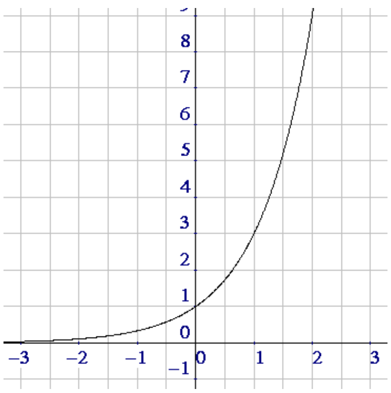
e. $t = 3$ f. 59,049 / 90 $\approx$ 656

2. a. 64 b. $\frac{1}{64}$ c. $\frac{1}{2}$ d. 8

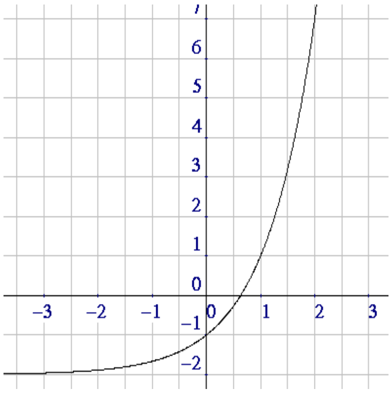
3. x can be all kinds of numbers, so the domain is probably $\mathbb{R}$.

4. If $4^x = 0$, we're saying that the graph of $y = 4^x$ has an x-intercept, which it doesn't. Therefore, $4^x = 0$ has no solution.

5.



6.



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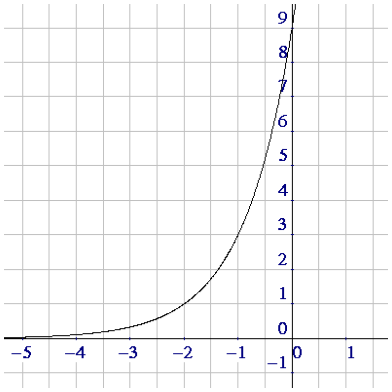

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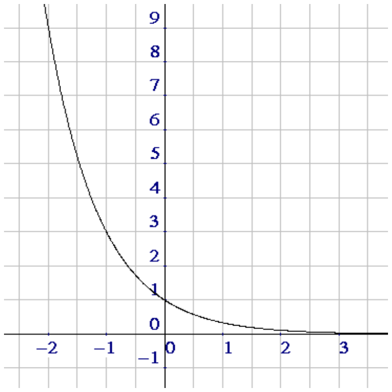


9. Same graph as #14, since $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

10. The base must be positive but not equal to 1. That is, the function $f(x) = b^x$ is an exponential function if $b > 0$, but $b \neq 1$. In other words, the base b must be in the set: $(0, \infty) - \{1\}$.

11. If we consider the exponential function $y = (-9)^x$, then we could not use $x = \frac{1}{2}$, since $y = (-9)^{\frac{1}{2}} = \sqrt{-9} \notin \mathbb{R}$.

12. 2/3, 0.987, π, 200

13. False

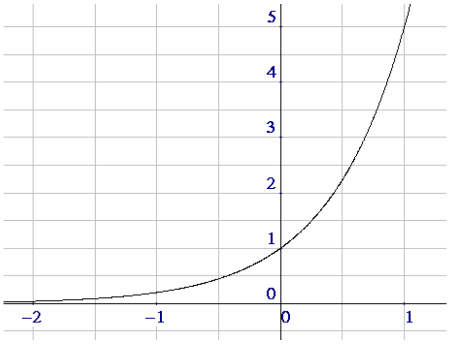
14. $\{x \in \mathbb{R} \mid x > 0, x \neq 1\}$ –OR– Any positive real number $\neq 1$

–OR– $(0, 1) \cup (1, \infty)$ –OR– $(0, \infty) - \{1\}$

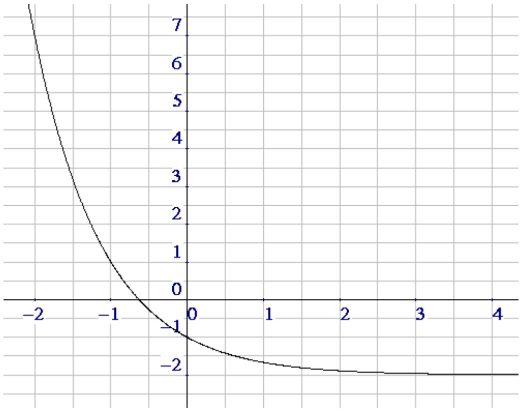
15. If b were 1, then $f(x) = b^x$ would become $f(x) = 1^x = 1$, which is a simple constant function whose graph is a horizontal line, rather useless to describe exponential growth and decay.

16. Ain’t no such animal

17.



18.



Horizontal asymptote: $y = -2$

19. $g(x) = 2^{x+7} + 4$

20. T **21.** F (any positive real number ≠ 1)

22. $\mathbb{R}$ **23.** horiz: $y = 0$; vert: none

24. Because $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

25. The graph of f is the graph of y shifted 2 units to the right and 4 units up.

26. F

27. Because if $f(x) = 0^x$, then $f(x) = 0$, which is just a horizontal line. Even a better reason: If $f(x) = 0^x$ and we choose $x = -4$, we get an output of 0^{-4}, which is $1/0^4$, which is 1/0, which is verboten!

28. a. F b. T c. F d. T e. T f. F
g. F h. F i. F j. F k. T l. F

The most beautiful experience we
can have is the mysterious.
It is the fundamental emotion
which stands at the cradle
of true art and true science.”

Albert Einstein